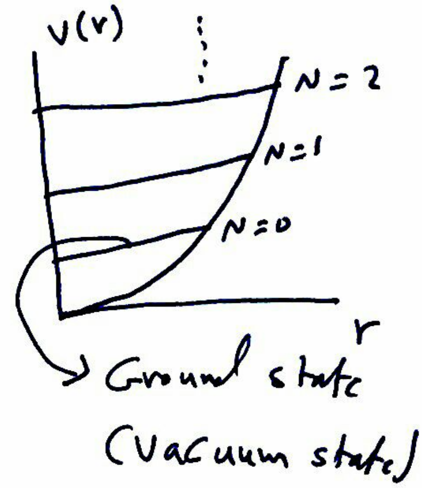
Section 6.3.4: The isotropic Harmonic oscillator

solving the radial equation of the isotropic H.O will be left as an excersize. However, I will discuss an easier approach making use of what we know about 1D H.O



Now $V(r) = \frac{1}{2} m\omega^2 r^2$

$$= \frac{1}{2} m\omega^2 (x^2+y^2+z^2)$$

$$= \frac{1}{2} m\omega^2 x^2 + \frac{1}{2} m\omega^2 y^2 + \frac{1}{2} m\omega^2 z^2$$

$$= V_x(x) + V_y(y) + V_z(z)$$

H can be written as a sum of 3 independent terms

$$H = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2 + \frac{p_y^2}{2m} + \frac{1}{2} m\omega^2 y^2 + \frac{p_z^2}{2m} + \frac{1}{2} m\omega^2 z^2$$

$$= H_x + H_y + H_z$$

$\Rightarrow$ the solution is then can be written as a product of the individual wave functions along x, y, and z as

$$\psi(r) = \psi_{n_x}(x)\, \psi_{n_y}(y)\, \psi_{n_z}(z),$$

where $\psi_{n_x}(x) = C_{n_x} \underbrace{H_{n_x}(\xi)}_{\text{Hermite polynomials}} e^{-\xi^2/2}$; $\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$

with $H_0(x) = 1$; $H_1(x) = x$; $H_2(x) = 1 - 2x^2$

$$\therefore \psi(r) = C_{n_x} C_{n_y} C_{n_z} H_{n_x}\left(\sqrt{\tfrac{m\omega}{\hbar}}\, x\right) H_{n_y}\left(\sqrt{\tfrac{m\omega}{\hbar}}\, y\right) H_{n_z}\left(\sqrt{\tfrac{m\omega}{\hbar}}\, z\right) e^{-\frac{m\omega}{2\hbar} r^2}$$

so for Ground state $(n_x = n_y = n_z = 0)$, we have

$\therefore \psi_{n_x,n_y,n_z}(r) = \psi_{0,0,0}(r) = C_0^3 e^{-\frac{m\omega}{2\hbar} r^2}$ unique state

C_0^3 can be found from normalization as follows

$$\int_{-\infty}^{\infty} d^3r \, |\psi_{000}|^2 = 1$$

$$(C_0^3)^2 \int_0^\infty dr \, r^2 e^{-\frac{m\omega}{\hbar} r^2} \underbrace{\int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi}_{4\pi} = 1$$

$\rightarrow$ solid angle $\int d\Omega = 4\pi$

$$4\pi (C_0^3) \underbrace{\int_0^\infty dr \, r^2 e^{-\frac{m\omega}{\hbar} r^2}}_{\frac{1}{2} \frac{1}{2\frac{m\omega}{\hbar}} \sqrt{\frac{\pi}{(m\omega/\hbar)}}} = 1$$

$\rightarrow$ Gaussian integral using $\int_0^\infty dx \, x^2 e^{-ax^2} = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$

$$(C_0^3)^2 \left(\frac{\hbar\pi}{m\omega}\right)^{3/2} = 1 \quad \Rightarrow \quad C_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4}$$

$$\Rightarrow \psi_{000}(r) = \left(\frac{m\omega}{\pi\hbar}\right)^{3/4} e^{-\frac{m\omega}{2\hbar} r^2} \text{ as expected}$$

Energy of the 3D isotropic H.O is

$$E = \hbar\omega(n_x + 1/2 + n_y + 1/2 + n_z + 1/2) = \hbar\omega(n_x + n_y + n_z + 3/2)$$

$$= \hbar\omega(N + 3/2) \; ; \text{ where } N = n_x + n_y + n_z$$

for Ground state $N = 0 = n_x + n_y + n_z$, $E = \frac{3}{2}\hbar\omega$

notice that the Ground state is Non degenerate.

the degeneracy of the state N is

$$g_N = \frac{1}{2}(N+1)(N+2)$$

$$g_0 = 1 \longrightarrow \text{unique}$$

Now for first excited state $N = 1 = n_x + n_y + n_z$

$E = \frac{5}{2}\hbar\omega$ and we have 3 degenerate states

i.e $(n_x, n_y, n_z) = (1,0,0) ; (0,1,0), (0,0,1)$

when

$$\psi_{100} = \psi_1(x)\,\psi_0(y)\,\psi_0(z) = C_0^2 C_1\, H_1(x)\, H_0(y)\, H_0(z)\, e^{-\frac{m\omega}{2\hbar} r^2}$$

$$= C_0^2 C_1\, x\, e^{-\frac{m\omega}{2\hbar} r^2}$$

$$\psi_{010} = \psi_0(x)\,\psi_1(y)\,\psi_0(z) = C_0^2 C_1\, H_0(x)\, H_1(y)\, H_0(z)\, e^{-\frac{m\omega}{2\hbar} r^2}$$

$$= C_0^2 C_1\, y\, e^{-\frac{m\omega}{2\hbar} r^2}$$

$$\psi_{001} = \psi_0(x)\,\psi_0(y)\,\psi_1(z) = C_0^2 C_1\, z\, e^{-\frac{m\omega}{2\hbar} r^2}$$

- for 2^{nd} excited state $N = 2 = n_x + n_y + n_z$

and we have six states $(g_2 = 6)$

$E = \frac{7}{2}\hbar\omega$

$(200)(020)(002)(110)(101)(011)$

$$\psi_{200} \sim (1 - 2x^2)\, e^{-\frac{m\omega}{2\hbar} r^2}$$

$$\psi_{020} \sim (1 - 2y^2)\, e^{-\frac{m\omega}{2\hbar} r^2}$$

$$\psi_{002} \sim (1 - 2z^2)\, e^{-\frac{m\omega}{2\hbar} r^2}$$

$$\psi_{110} \sim xy\, e^{-\frac{m\omega}{2\hbar} r^2}$$

$$\psi_{101} \sim xz\, e^{-\frac{m\omega}{2\hbar} r^2}$$

$$\psi_{011} \sim yz\, e^{-\frac{m\omega}{2\hbar} r^2}$$